The Worldsheet Formulation as an Alternative Method for Simulating Dynamical Fermions.

Hugo Fort
Instituto de Física, Facultad de Ciencias,
Tristan Narvaja 1674, 11200 Montevideo, Uruguay

The recently proposed worldsheet formulation of lattice fermions is tested for the first time carrying out a simulation for the simplest model: the one-flavor, strictly massless lattice Schwinger model. A main advantage of this alternative method for simulating dynamical fermions consists in its economy: it involves many fewer degrees of freedom than the ordinary Kogut-Susskind formulation. The known continuum limit is reproduced by the method for relatively small lattices.

It is well known that a complete description of Quantum Chromodynamics (QCD) requires a non-perturbative approach. The main available non-perturbative method is lattice gauge theory. In particular, the *Lagrangian formulation*, with discretized Euclidean time, allows to apply the powerful numerical method of Monte Carlo simulation. However, simulating QCD with dynamical fermions or *full* QCD is still too expensive in computer time. Throuhout the years the conclusion has been the same: either more machine power and/or a real improvement in algorithms is needed to produce reliable estimates of hadronic quantities [1].

The root of the problems of QCD can be traced to the fact that the theory is defined in terms of local fields, quarks and gluons, but the physical excitations are extended composites: mesons and baryons. A Hamiltonian lattice formulation directly in terms of string-like excitations gave rise to the so-called *P-representation* [2], [3]. The states $|P>$ of this representation are described by sets of lattice paths associated to the extended excitations. We propose to explore a new numerical approach based on the recently introduced worldsheet formulation of lattice gauge theories with dynamical fermions [4]. This formulation, in terms of the worldsheets of the extended excitations, is the Lagrangian counterpart of the P-representation. The worldsheet partition function of lattice QED with staggered fermions is expressed as a sum over surfaces with border on self-avoiding fermionic loops. Each surface is weighted with a classical action written in terms of integer gauge invariant variables. The surfaces correspond to the worldsheets of loop-like pure electric flux excitations and meson-like configurations (open electric flux tubes carrying matter fields at their ends). This description, besides the general advantage of geometric transparency, is appealing because it involves fewer degrees of freedom than the ordinary Kogut-Susskind formulation implying a substantial economy of CPU time. Furthermore, the present formulation of dynamical fermions in terms of self-avoiding loops is closely connected with the polymer-like representation [6] and this might enable to exploit techniques used in other branches of physics, like condensed matter physics [7] and cosmic string physics [8].

In this letter we present, for the first time, a test of the formalism developed in [4]. The simplest lattice gauge theory with dynamical fermions, the Schwinger model or (1+1) QED, is chosen and a Monte Carlo simulation is performed. This massless model can be exactly solved in the continuum. However, it is rich enough to share many features with 4-dimensional QCD. For this reason it has been extensively used as a laboratory to analyze the previous phenomena and has also become a popular benchmark to test different techniques to simulate theories with fermions. Our main point is to show that the method works properly and it might provide a novel alternative approach to simulate *full* QCD.

The P-representation offers a gauge invariant description of physical states in terms of kets $\mid P >$, where P labels a set of connected paths P_x^y oriented from the even sites x to the odd ones y in a lattice of spacing a. These paths correspond to string-like "electromeson" excitations of staggered fermions connected by tubes of electric flux. The internal product of a ket $\mid P_x^y >$ with one in terms of fields is given by

$$\Phi(P_x^y) \equiv < P_x^y \mid \psi_u^\dagger, \psi_d, \theta_\mu >= \psi_u^\dagger(x) U(P_x^y) \psi_d(y), \tag{1}$$

where $U_\mu(x) \equiv \exp[i\theta_\mu(x)] \equiv \exp[ieaA_\mu(x)]$ and u and d denote respectively the up and down parts of

the Dirac spinor. Therefore, the lattice paths P_x^y start in sites x of a given parity and end in sites y with opposite parity. The one spinor component at each site can be described in terms of the Susskind's $\chi(x)$ single Grassmann fields [5]. The path creation operator $\hat{\Phi}_Q$ in the space of kets $\{|\, P>\}$ of a path with ends x and y is defined as

$$\hat{\Phi}_Q = \hat{\chi}^\dagger(x)\hat{U}(Q_x^y)\hat{\chi}(y). \tag{2}$$

Its adjoint operator $\hat{\Phi}_Q^\dagger$ acts in two possible ways [2]: annihilating the path Q_x^y or joining two existing paths in $|\, P>$ one ending at x and the other starting at y.

The worldsheet partition function Z_P is a sum over the worldsheets of strings or paths of the P-representation. That is, surfaces $S_{\mathcal{F}^c}$ such that: **(I)** their borders $\mathcal{F}^c$ are self-avoiding polymer-like loops and **(II)** when intersected with a time t = constant plane they produce paths beginning at even sites and ending at odd ones. This description at first sight is similar to the one obtained by integrating the fields in the Kogut-Susskind partition function Z_{KS}, nevertheless it differs in two features. In first place, the integration of the fields in Z_{KS} produces, besides surfaces bounded by fermionic loops, isolated links traversed in both opposite directions or "null" links. These "null" links were ruled out from the Z_P and with them the myriad of different configurations corresponding to a given configuration of worldsheets. In second place, by virtue of the constraint **(II)**, the surfaces of Z_P when intersected with a time t = constant plane produce only paths beginning at even sites and ending at odd ones instead of the paths having ends of any parity (as it happens with the surfaces obtained from Z_{KS}). These two differences translate respectively in: computer time economy and a cure for the additional species doubling problem of the Kogut-Susskind action. Concerning the last point, the worldsheet action has only 2 fermion species in 4 dimensions (instead of 4 fermion species) which is phenomenologically more satisfactory. In ref. [4] it was proved that Z_P leads to the QED Hamiltonian using the transfer matrix procedure. The expression for Z_P is as follows [9]:

$$Z_P = \sum_{S_{\mathcal{F}^c}} \sigma(\mathcal{F}^c)\, \exp\{-\frac{1}{2\beta}\sum_{p\in S_{\mathcal{F}^c}} n_p^2\}, \tag{3}$$

where n_p is an integer variable attached to plaquettes (a 2-form) and $\sigma_{\mathcal{F}}$ is a sign given in terms of purely geometric quantities of the fermionic loops $\mathcal{F}^c$ [4]. In $D=2$, it turns out that $\sigma(\mathcal{F}^c) = (-1)^{N_{\mathcal{F}^c} - \frac{L_{\mathcal{F}^c}}{2} + A_{\mathcal{F}^c}}$ – where $N_{\mathcal{F}^c}$, $L_{\mathcal{F}^c}$ and $A_{\mathcal{F}^c}$ are, respectively, the number of connected parts, the length and the area of $\mathcal{F}^c$ – and all the non-vanishing contributions have $\sigma_{\mathcal{F}} = +1$. The reason is that $N_{\mathcal{F}^c} - \frac{L_{\mathcal{F}^c}}{2} + A_{\mathcal{F}^c} = I_{\mathcal{F}}^c$, the number of enclosed sites by the fermionic loops $\mathcal{F}^c$ which is always even by virtue of the above constraints **(I)** and **(II)** (see ref. [4] for more details), so that we omit this factor.

The fermionic paths $\mathcal{F}^c$ can be expressed in terms of integer 1-forms –attached to the links– f with three possible values: 0 and $\pm$ 1 with the constraint that they are non self-crossing and closed ($\partial f = 0$) as

$$Z_P^{\text{Schwinger}} = \sum_n \sum_f \exp\{-\frac{1}{2\beta}\|n\|^2\}\delta(f - \partial n). \tag{4}$$

The lattice chiral condensate per-lattice-site is defined as $<\bar{\chi}\chi> = \frac{1}{2N_s}\sum_x (-1)^{x_1} < [\hat{\chi}^\dagger(\mathrm{x}), \hat{\chi}(\,\mathrm{x})] >$, where N_s is the number of lattice sites. The corresponding operator is realized in the P-representation and thus we get for the chiral condensate [2]:

$$<\bar{\chi}\chi> = \frac{1}{2} - \frac{2\mathcal{N}_P}{N_s}, \tag{5}$$

where $\mathcal{N}_P$ is the number of connected paths at a given time t. Thus, equation (5) allows to calculate directly the chiral condensate simply by counting the number of "electromesons" we have when we intersect their world sheets with each time slice t.

To generate the worldsheets we use a Metropolis-type updating algorithm with the Boltzmann weight proportional to $\exp\{-\frac{1}{2\beta}\sum_p n_p^2\}$. We simulate the model on $L\times L$ square lattices with periodic boundary conditions (*pbc*). All the plaquettes p belonging to a surface, by virtue of the self-avoiding constraint **(I)** on their borders, must fulfill the condition that the difference between the integers n_p of contiguous plaquettes has 3 possibilities: $\Delta_\mu n_p = \pm 1$ or 0 ($\mu = 1, 2$). On the other hand, the constraint **(II)** implies that $\Delta_1 n_p = 1 \text{or} -1$ according to the parity of the spatial coordinate x_1. With these simple rules the algorithm generates the surfaces. In the computation of the chiral condensate per-lattice-site $< \bar{\chi}\chi >$ there is a clear difference between a simulation in the ordinary representation, in terms of fields, and one in the P-representation. In the first case, for the massless model, given enough time, the system rotates through all the degenerate minima so that $< \bar{\chi}\chi >= 0$. Therefore, one has to calculate this order parameter for several small masses $m \neq 0$, which select the $\theta = 0$ vacuum, and then extrapolate to the limit $m \to 0$. On the other hand, in the gauge invariant P-representation from the very beginning $\theta = 0$ [10] and the computation with $m = 0$ can be performed directly. To compute $< \bar{\chi}\chi >$ using (5) we have to count $\mathcal{N}_P$ at each one of the L time slices. Therefore, for each lattice sweep we collect L values of $< \bar{\chi}\chi >$. Typically, 100000 sweeps per point were performed.

The first thing we have checked is that we get the right strong coupling behavior, both for the ground-state energy density ω_0 and the chiral condensate per-lattice-site $< \bar{\chi}\chi >$. The agreement with the series expansion is pretty good up to $\beta = 0.12$.

On the other hand, the continuum theory is reached at zero lattice coupling, in the same way as four-dimensional asymptotically free theories like QCD. It is known that the chiral condensate in the continuum is given exactly by

$$\frac{< \bar{\psi}\psi >}{e} = \frac{e^\gamma}{2\pi^{3/2}} = 0.15995 \qquad (\gamma \text{ is Euler's constant}). \tag{6}$$

As the lattice size L increase, the convergence of $q \equiv < \bar{\chi}\chi > /e$ to its known continuum value (6) improves. A property of the action of (4) is that q becomes stabilized at its continuum value well inside the weak coupling. Before reaching this regime, this observable oscillates strongly with β. In FIG.1 we plot q vs. β for lattices of sizes ranging from $L = 20$ to $L = 32$. For $L = 32$ it is apparent that a value close to the exact continuum one is reached.

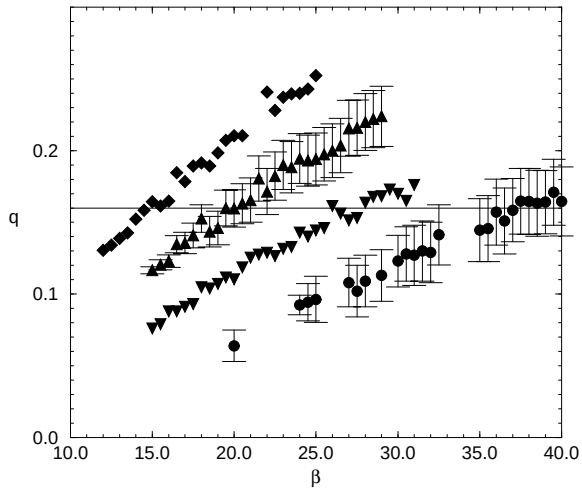

FIG. 1. $q \equiv < \bar{\chi}\chi > /e$ for different lattice sizes $L = 20$ (diamonds), $L = 24$ (triangles up) , $L = 28$ (triangles down) and $L = 32$ (circles). The horizontal line correspond to the known continuum value $\simeq 0.16$ (statistical errors are indicated by vertical bars for $L = 24$ and $L = 32$).

Our aim has been to show that the recently introduced worldsheet formulation is a valuable alternative tool in order to do numerical computations with dynamical fermions. So, at this stage, we have chosen the simplest algorithm: the standard Metropolis one. Obviously, there are several more sophisticated techniques that can be applied in order to improve the results. Additionally, the code can be, of course, optimized.

The method presents the following advantages: **1)** Easiness of computation. For instance, the lattice chiral condensate is diagonal in the P-representation and hence to evaluate it one simply has to count the number of connected open string-like excitations. Furthermore, one can compute directly $< \bar{\chi}\chi >$ for the case of *strictly 0 mass*. **2)** Economy I: it involves much less degrees of freedom that the Kogut-Susskind action. **3)** Economy II: no gauge redundancy. **4)** As a consequence of the constraint **(II)**, the worldsheet action does not suffer from the additional species doubling problem of the Kogut-Susskind action.

Concerning the results: we get the right continuum limit (the weak coupling fixed point) as one can see from the chiral condensate which is consistent with the exact results for the continuum theory. This is a numerical confirmation of the redundancy of the information carried out by the "null" links and the correctness of their elimination from the partition function.

Finally, we would like to stress (once more) that our aim was not to present another solution to the Schwinger model, but to test a new general approach to tackle dynamical fermions. The results are promising, and this encourages us to employ more refined numerical methods in a following stage. To extend the method to $D > 2$ one has to compute the sign σ of each configuration. We are working on an algorithm to compute efficently this factor.

Acknowledgements

I am indebted with J. M. Aroca and R. Gambini for many valuable discussions. I also would like to thank V. Azcoiti, M. Baig and G. Beltrame for helpful technical comments.

This work was supported in part by CONICYT, Project No. 318.

---

[1] QCD-TARO Collaboration, *Lattice 96*, Nucl. Phys. **B53** (Proc. Suppl.) (1997) 938; Philippe de Forcrand, *Lattice 95* Published in Lattice 1995 (1996) 228; C. Michael, *NATO Advanced Study Institute on Hadron Spectroscopy and the Confinement Problem* (1995); C. Michael, *Lattice 94*, Nucl. Phys. **B42** (Proc. Suppl.) (1995); D. Weingarten, *Lattice 93*, Nucl. Phys. **B34** (Proc. Suppl.) (1994) 29; A. Ukawa, *Lattice 92*, Nucl. Phys. **B30** (Proc. Suppl.) (1993) 3 ;D. Toussaint, *Lattice 91*, Nucl. Phys. **B26** (Proc. Suppl.) (1992) 3.

[2] H. Fort and R.Gambini, Phys. Rev. **D44** (1991) 1257.

[3] R. Gambini and L. Setaro, Nucl. Phys. **B448** (1995) 67.

[4] J.M. Aroca, H. Fort and R. Gambini, hep-lat/9607050.

[5] L. Susskind, Phys. Rev. **D16** (1977) 3031; Banks T., D.R.T. Jones, J. Kogut, S. Raby, P.N. Scharbach, D.K. Sinclair and L. Susskind, Phys. Rev. **D15** (1977) 1111.

[6] I. Montvay and G. Münster, *Quantum Fields on the Lattice.* Cambridge University Press (1994).

[7] F.W. Wiegel, *Introduction to Path-Integral Methods in Physics and Polymer Science.* World Scientific, Singapore, 1986.

[8] E. Copeland, D. Haws and R. Rivers, Nucl. Phys. **B319** (1989) 687.

[9] This version of Z_P corresponds to the Villain form of the lattice action (J. Villain, J. Phys. (Paris) **36** (1975) 581). We chose this action only for simplicity.

[10] The P-Representation is a gauge invariant description which does not "see" the *large* gauge transformations.